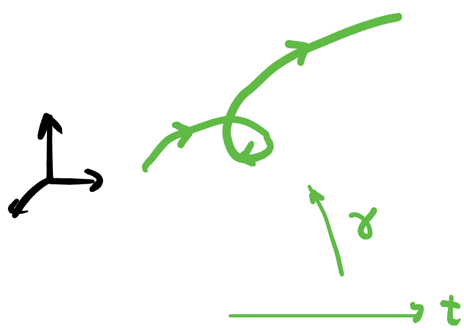
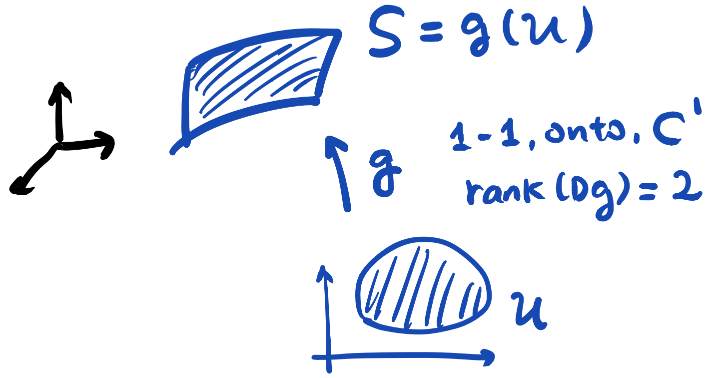
MATH 2028 Integration on submanifolds of $\mathbb{R}^n$

Goal: Define k-dimensional submanifolds of $\mathbb{R}^n$ and discuss how to do integration on them.

Recall: Curves in $\mathbb{R}^n$ Surfaces in $\mathbb{R}^3$





We first generalize it to any dimension.

Def$^{\text{n}}$: A k-dimensional parametrized submanifold of $\mathbb{R}^n$ is a C^1 map $g: \mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^k \longrightarrow \mathbb{R}^n$ s.t.

- g is 1-1, onto its image $S = g(\mathcal{U})$
- $\operatorname{rank}(Dg) = k$ everywhere in $\mathcal{U}$

Any such g is called a parametrization of S.

We can piece together such parametrized submanifolds to form a global submanifold, which may not be expressed globally by a single parametrization (e.g. spheres).

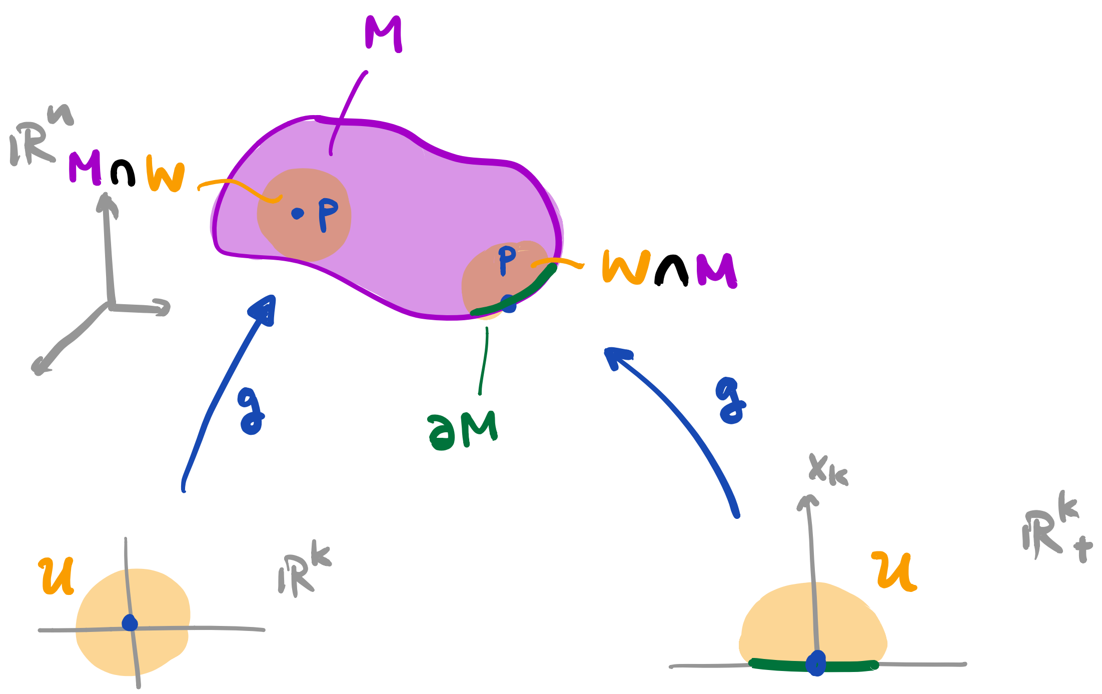
Def^n: A subset $M \subseteq \mathbb{R}^n$ is called a **k-dimensional submanifold of $\mathbb{R}^n$ with boundary** if $\forall p \in M$,

$\exists\ W \overset{\text{open}}{\subseteq} \mathbb{R}^n$ containing p and a parametrization

$g : \mathcal{U} \to \mathbb{R}^n$ s.t.

(i) $g(\mathcal{U}) = W \cap M$

(ii) $\mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^k$ or $\mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^k_+ := \{ x_k \geq 0 \}$



Note that the tangent space T_pM at $p \in M$ is spanned by the basis (denote $g = g(u_1, \ldots, u_k)$)

$$\left\{ \frac{\partial g}{\partial u_1}, \frac{\partial g}{\partial u_2}, \ldots, \frac{\partial g}{\partial u_k} \right\} \subseteq T_pM$$

Using this basis, we can define

$$g_{ij} := \frac{\partial g}{\partial u_i} \cdot \frac{\partial g}{\partial u_j} . \quad i,j = 1, \dots, k$$

Denote (g_{ij}) to be the $k \times k$ matrix.

<u>Def$^{\underline{n}}$</u>: For any cts function $f: M \to \mathbb{R}$ on a k-dimensional submanifold $M \subseteq \mathbb{R}^n$ parametrized by $g: \mathcal{U} \to M = g(\mathcal{U})$, we define the integral of f as

$$\int_M f \, d\sigma := \int_{\mathcal{U}} f \circ g \cdot \sqrt{\det(g_{ij})} \, dV$$

In particular, $\text{Area}(M) := \int_M 1 \, d\sigma$.

<u>Example</u>: Let $M \subseteq \mathbb{R}^4$ be the 2-dim'l submanifold parametrized by $g: (0, 2\pi) \times (0, 2\pi) \longrightarrow \mathbb{R}^4$

$$g(u,v) = \left(\tfrac{1}{\sqrt{2}} \cos u , \tfrac{1}{\sqrt{2}} \sin u , \tfrac{1}{\sqrt{2}} \cos v , \tfrac{1}{\sqrt{2}} \sin v \right)$$

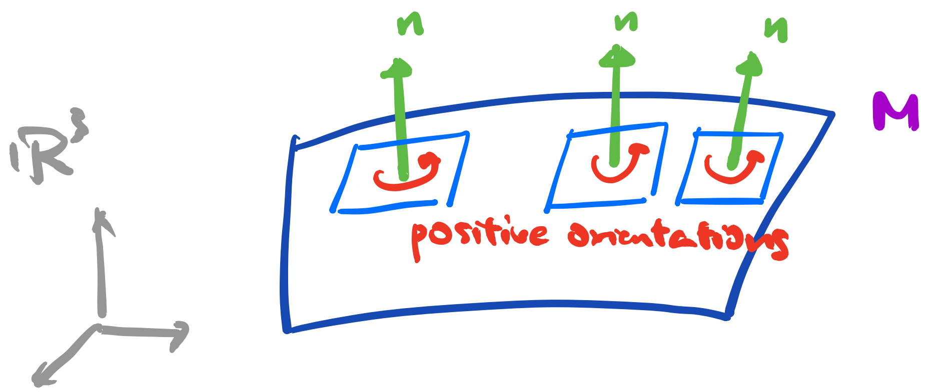
Then, the area of M is given by

$$\text{Area}(M) = \int_0^{2\pi} \int_0^{2\pi} \sqrt{\det \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}} \, du \, dv = 2\pi^2$$

$\blacksquare$

An orientation on M is a continuously choice of "positively" oriented basis on each T_pM. If such choice exists, we say M is orientable.

For surfaces $M \subseteq \mathbb{R}^3$, this can be alternatively described by a globally defined unit normal n together with the "right hand rule".



FACT: M orientable $\Leftrightarrow$ $\exists$ nowhere vanishing k-form on M

Def^n: Let $M \subseteq \mathbb{R}^n$ be an oriented k-submanifold. The volume form of M is a k-form σ with the property that

$$\sigma(p)(\underbrace{v_1, \ldots, v_k}_{\text{vectors in } T_pM}) = \text{signed volume of parallelopiped spanned by } v_1, \ldots, v_k$$

We now move on to discuss how to integrate k-forms on k-dimensional submanifolds $M \subseteq \mathbb{R}^n$.

For an n-form $\omega = f(x)\, dx_1 \wedge \cdots \wedge dx_n$ in a domain $\Omega \subseteq \mathbb{R}^n$, we define:

$$\int_\Omega \omega := \int_\Omega f \, dV$$

Then, the Change of Variable Theorem can be expressed as: $g : \Omega \subseteq \mathbb{R}^n \to g(\Omega) = S \subseteq \mathbb{R}^n$

$$\boxed{\int_S \omega = \int_\Omega g^* \omega}$$

where

$\omega = f(x)\, dx_1 \wedge \cdots \wedge dx_n$

$$\left(\begin{aligned} \underline{\text{Reason}}: \; & g^*(dx_1 \wedge \cdots \wedge dx_n) \\ &= \det(Dg)\, dx_1 \wedge \cdots \wedge dx_n \end{aligned} \right)$$

<u>Def$^{\underline{n}}$</u>: Given a parametrization $g : \mathcal{U} \to \mathbb{R}^n$ of a k-dim'l submanifold $M = g(\mathcal{U})$ and a k-form ω in $\mathbb{R}^n$, we define

$$\boxed{\int_M \omega := \int_{\mathcal{U}} g^* \omega}$$

Integral of k-forms on k-submanifolds

Consequence: Suppose $g_1 : \mathcal{U}_1 \to \mathbb{R}^n$, $g_2 : \mathcal{U}_2 \to \mathbb{R}^n$ are parametrizations of the same k-submanifold M, then by Change of Variable Theorem.

$$\int_{\mathcal{U}_1} g_1^* \omega = \int_{\mathcal{U}_2} g_2^* \omega$$

Hence, the definition above is independent on the choice of parametrization. From this, we can define the integral of k-forms on a k-dim'l submanifold which is not necessarily covered by one parametrization.

The "trick" is again "Partition of unity"

Fact: Let $M \subseteq \mathbb{R}^n$ be a compact k-dim'l submfd with boundary. THEN, $\exists$ smooth functions

$$\rho_i : M \to [0,1], \quad i = 1, \dots, N$$

each ρ_i is supported in some parametrization

and $\displaystyle\sum_{i=1}^{N} \rho_i(x) = 1 \quad \forall x \in M$

Def$^{\underline{n}}$: Let $M \subseteq \mathbb{R}^n$ be a compact, oriented, k-dim'l submanifold with boundary, and ω be a k-form on $\mathbb{R}^n$. Suppose $\{\rho_i\}_{i=1}^N$ is a partition of unity as above and $\operatorname{spt}(\rho_i)$ is contained in the parametrization $g_i : \mathcal{U}_i \to \mathbb{R}^n$ which is "orientation-preserving". THEN, we define

$$\int_M \omega := \sum_{i=1}^{N} \int_{g_i(\mathcal{U}_i)} \rho_i \omega$$